I. BÜHRING'S ANALYTIC CONTINUATION SERIES WHEN $a-b$ IS AN INTEGER

Without a loss of generality assume that the difference between the a and b parameters of Gauss's hypergeometric function $F(a,b,c;z)$ is a positive integer s or zero, i.e., assume that:

$$a-b=s, \quad s=0,1,2,3,...$$

Then equation (5) in [1] can be re-written as

$$F(a,b,c;z)=V_1(z)+(z_0-z)^{-a}\sum_{n=0}^{\infty}H_n(z_0)\,(z-z_0)^{-n} \tag{1}$$

where $V_1(z)$ is the finite expansion given by

$$V_1(z)=\frac{\Gamma(c)\,(s-1)!}{\Gamma(a)\,\Gamma(c-b)}\,(z_0-z)^{-b}\sum_{n=0}^{s-1}\frac{(b)_n}{(1-s)_n}\,e_n^{(b)}\,(z-z_0)^{-n} \tag{2}$$

with the convention that $V_1(z)=0$ when $s=0$. The coefficients $H_n(z_0)$ in eq. 1 are given as:

$$\begin{aligned}H_n(z_0)=&\frac{(-1)^{s+1}\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-a)}\,\frac{(a)_n}{(s+n)!}\,\left[\left(\tau_n^{(a)}-\ln(z_0-z)\right)\,e_n^{(a)}+f_n^{(a)}\right]\\&+\frac{\Gamma(c)\,(b)_{n+s}}{\Gamma(a)\,\Gamma(c-b)}\,\left[\tau_n^{(b)}\,e_{n+s}^{(b)}+f_{n+s}^{(b)}\right]\end{aligned} \tag{3}$$

where the $\tau_n^{(a,b)}$ variables can be expressed in terms of the digamma function ψ:

$$\begin{aligned}\tau_n^{(a)}&=\psi(c-a)+\psi(a+n)-\psi(a)-\psi(n+s+1)\\\tau_n^{(b)}&=\psi(n+1)-\psi(a)\end{aligned}$$

The $e_n^{(a,b)}$ and $f_n^{(a,b)}$ terms that appear in eqs. 2 and 3 are determined from the following four recurrence relations ($n\geq 1$),

$$\begin{aligned}e_n^{(a)}&=\frac{1}{n}\,\left[((n+a)(1-2z_0)+(a+b+1)z_0-c)\;e_{n-1}^{(a)}+z_0(1-z_0)(n-1+s)\,e_{n-2}^{(a)}\right]\\e_n^{(b)}&=\frac{1}{n}\,\left[((n+b)(1-2z_0)+(a+b+1)z_0-c)\;e_{n-1}^{(b)}+z_0(1-z_0)(n-1-s)\,e_{n-2}^{(b)}\right]\\f_n^{(a)}&=\frac{1}{n}\,\Big[(1-z_0)\;e_{n-1}^{(a)}+z_0\,(1-z_0)\;e_{n-2}^{(a)}\\&\qquad+[(n+a)(1-2z_0)+(a+b+1)z_0-c]\;f_{n-1}^{(a)}+z_0(1-z_0)(n-1+s)\,f_{n-2}^{(a)}\Big]\\f_n^{(b)}&=\frac{1}{n}\,\Big[z_0\,e_{n-1}^{(b)}-z_0\,(1-z_0)\;e_{n-2}^{(b)}\\&\qquad+[(n+b)(1-2z_0)+(a+b+1)z_0-c]\;f_{n-1}^{(b)}+z_0(1-z_0)(n-1-s)\,f_{n-2}^{(b)}\Big]\end{aligned}$$

with the initial conditions:

$$
\begin{aligned}
&e_0^{(a)} = e_0^{(b)} = 1 \\
&e_{-1}^{(a)} = e_{-1}^{(b)} = f_0^{(a)} = f_0^{(b)} = f_{-1}^{(a)} = f_{-1}^{(b)} = 0
\end{aligned}
$$

These formula's will be implemented in hyp2f1 with $z_0 = 1/2$.

---

[1] W. Bühring, An Analytic Continuation of the Hypergeometric Series, SIAM J. Math. Anal., Vol 18, No 3, May 1987