Spectral Leakage

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1 Fourier series

Consider a continuous signal $x(t)$ measured over a time interval $[-T/2, T/2]$. This signal can be represented as a **Fourier series**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{i2\pi kt/T} \tag{1}$$

where

$$c_k \equiv \frac{1}{T}\int_{-T/2}^{T/2} dt\, x(t)\, e^{-i2\pi kt/T} . \tag{2}$$

The index k gives the frequency of each component in units of cycles of the signal over the measured interval. In other words, the term with $k = 1$ has frequency k/T. This is important to remember, the frequency resolution of a Fourier series is precisely the inverse of the measurement time T. We call the frequencies $\{k/T\}$ the **Fourier frequencies**.

A complex sinusoid with frequency equal to one of the Fourier frequencies has a delta function Fourier series. For example, the signal $s(t) = \exp[i2\pi lt/T]$ has Fourier series $c_k = \delta_{kl}$.

1.1 Shift in frequency

Consider a signal $s(t)$ with Fourier series coefficients s_k. Multiplying $s(t)$ by a complex exponential shifts the Fourier coefficients. If we construct a new signal $u(t) \equiv s(t)\exp[i2\pi lt/T]$, then the Fourier series is

$$\begin{aligned} u_k &= \int_{-T/2}^{T/2} s(t) e^{i2\pi lt/T} e^{-i2\pi kt/T} \\ &= \int_{-T/2}^{T/2} s(t) e^{-i2\pi(k-l)t/T} \\ &= c_{k-l} . \end{aligned} \tag{3}$$

2 Non-commensurate frequency

Consider a signal $x(t) = \exp[i2\pi\xi t/T]$ where ξ is an arbitrary real number. The Fourier series coefficients of this signal are

$$\begin{aligned} c_k &= \frac{1}{T}\int_{-T/2}^{T/2} dt\, e^{i2\pi\xi t/T} e^{-i2\pi kt/T} \\ &= \frac{1}{T}\left(\frac{\exp[i2\pi(\xi-k)t/T]}{i2\pi(\xi-k)/T}\right)\Bigg|_{-T/2}^{T/2} \\ &= \frac{1}{T}\left(\frac{\exp[i\pi(\xi-k)] - \exp[-i\pi(\xi-k)]}{i2\pi(\xi-k)/T}\right) \\ &= \frac{\sin(\pi(\xi-k))}{\pi(\xi-k)} . \end{aligned} \tag{4}$$

This function has several important properties. First, if ξ is an integer, then all $c_k = \delta_{\xi k}$. This is not surprising because if ξ is an integer then $x(t)$ precisely matches one of the component functions $\exp(i2\pi kt/T)$. This situation is illustrated by the black curve in Figure 1 where we have put $\xi = 0$. The underlying $\sin(\pi x)/(\pi x)$ function goes to zero at

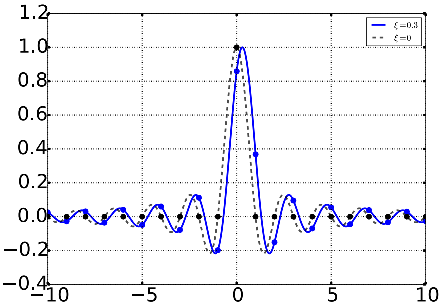

Figure 1: Fourier series of a complex sinusoid $\exp(i 2\pi \xi t/T)$ for integer and non-integer frequencies. Lines show the $\sin(\pi x)/(\pi x)$ function for $\xi = 0$ (black) and $\xi = 0.2$ (blue). The dots indicate the values of the curves at the Fourier frequencies.

every k except for $k = 0$. If ξ is not an integer then the $\sin(\pi x)/(\pi x)$ function shifts such that $c_k \neq 0$ for *all* k. This is illustrated by the blue curve in Figure 1 where we have used $\xi = 0.2$. Note that most of the amplitude of the signal sits near the real frequency ξ, but has leaked into neighboring frequency bins. This phenomenon, wherein a signal at a precise frequency ξ shows up with amplitude at other bins nearby is called *spectral leakage*. In summary, given a complex sinusoid at a frequency ξ not commensurate with the measurement window, the Fourier series leaks into the bins near ξ according to

3 Widows