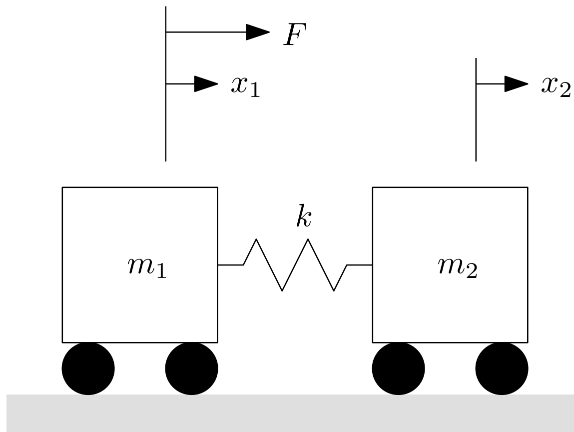

Figure 1: A two mass system used to introduce the transfer matrix method (TMM).

1 TMM Analysis

The transfer matrix for a mass is

$$Um = \begin{bmatrix} 1 & 0 & 0 \\ m\,s^2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

and for a spring

$$Us = \begin{bmatrix} 1 & \frac{1}{k} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

where the state vector is

$$z = \begin{bmatrix} x \\ F \\ 1 \end{bmatrix} \tag{3}$$

The two mass system shown in Figure 1 can be model as

$$\mathbf{z}_{\mathrm{end}} = \mathbf{U}_{\mathrm{m2}}\mathbf{U}_{\mathrm{s}}\mathbf{U}_{\mathrm{F}}\mathbf{U}_{\mathrm{m1}}\mathbf{z}_{\mathrm{start}} \tag{4}$$

where

$$Uf = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -F \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ m_1\,s^2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ m_2\, s^2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

and the system boundary conditions give that

$$zend = \begin{bmatrix} x_2 \\ 0 \\ 1 \end{bmatrix} \tag{8}$$

and

$$zstart = \begin{bmatrix} x_1 \\ 0 \\ 1 \end{bmatrix} \tag{9}$$

The system transfer matrix can be defined as

$$\mathbf{U}_{\text{sys}} = \mathbf{U}_{\text{m2}}\mathbf{U}_{\text{s}}\mathbf{U}_{\text{F}}\mathbf{U}_{\text{m1}} \tag{10}$$

or

$$Usys = \begin{bmatrix} \frac{m_1\, s^2}{k} + 1 & \frac{1}{k} & -\frac{F}{k} \\ m_2\, s^2 \left(\frac{m_1\, s^2}{k} + 1 \right) + m_1\, s^2 & \frac{m_2\, s^2}{k} + 1 & -\frac{m_2\, s^2\, F}{k} - F \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

and equation (4) can be rewritten as

$$\mathbf{z}_{\text{end}} = \mathbf{U}_{\text{sys}}\mathbf{z}_{\text{start}} \tag{12}$$

or

$$\begin{bmatrix} x_2 \\ 0 \\ 1 \end{bmatrix} = Usys \cdot \begin{bmatrix} x_1 \\ 0 \\ 1 \end{bmatrix} \tag{13}$$

The second row of this equations gives

$$0 = -\frac{m_2\, s^2\, F}{k} - F + \left(m_2\, s^2 \left(\frac{m_1\, s^2}{k} + 1 \right) + m_1\, s^2 \right) x_1 \tag{14}$$

which can be solved for the open-loop transfer function

$$tf1_T MM = \frac{x_1}{F} = \frac{m_2\, s^2 + k}{m_1\, m_2\, s^4 + (k\, m_2 + k\, m_1)\, s^2} \tag{15}$$

Solving equation (15) for x_1 and substituting the result into equation (13), allows the transfer function x_2/F to be found:

$$tf2_T MM = \frac{k}{m_1\, m_2\, s^4 + (k\, m_2 + k\, m_1)\, s^2} \tag{16}$$

1.1 Closed-Loop Analysis

1.1.1 Collocated Feedback

Proportional, collocated feeback control would take the form

$$F = g\,(u - x_1) \tag{17}$$

A transfer matrix model of the collocated feedback control would require the transfer matrix

$$Ucl1 = \begin{bmatrix} 1 & 0 & 0 \\ g & 1 & -g\,u \\ 0 & 0 & 1 \end{bmatrix} \tag{18}$$

and the system model would become

$$\mathbf{z}_{\mathrm{end}} = \mathbf{U}_{\mathrm{m2}}\mathbf{U}_{\mathrm{s}}\mathbf{U}_{\mathrm{cl1}}\mathbf{U}_{\mathrm{m1}}\mathbf{z}_{\mathrm{start}} \tag{19}$$

$$Usyscl = \begin{bmatrix} \frac{m_1\,s^2 + g}{k} + 1 & \frac{1}{k} & -\frac{g\,u}{k} \\ m_2\,s^2\,\left(\frac{m_1\,s^2 + g}{k} + 1\right) + m_1\,s^2 + g & \frac{m_2\,s^2}{k} + 1 & -\frac{g\,m_2\,s^2\,u}{k} - g\,u \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

The second row of this equations gives

$$0 = \left(m_2\,s^2\,\left(\frac{m_1\,s^2 + g}{k} + 1\right) + m_1\,s^2 + g\right)\,x_1 - \frac{g\,m_2\,s^2\,u}{k} - g\,u \tag{21}$$

which can be solved for the closed-loop transfer function

$$tf1cl_TMM = \frac{x_1}{u} = \frac{g\,m_2\,s^2 + g\,k}{m_1\,m_2\,s^4 + ((k+g)\,m_2 + k\,m_1)\,s^2 + g\,k} \tag{22}$$

2 Classical Analysis

The equation of motion for mass 1 is

$$m_1\,s^2\,x_1 = F - k\,(x_1 - x_2) \tag{23}$$

and for mass 2

$$m_2\,s^2\,x_2 = k\,(x_1 - x_2) \tag{24}$$

Equations (23) and (24) can be solved for the transfer functions

$$tf1 = \frac{m_2\,s^2 + k}{m_1\,m_2\,s^4 + (k\,m_2 + k\,m_1)\,s^2} \tag{25}$$

and

$$tf2 = \frac{k}{m_1\,m_2\,s^4 + (k\,m_2 + k\,m_1)\,s^2} \tag{26}$$

2.1 Collocated Feedback

For collocated feedback

$$tf1cl = \frac{x_1}{u} = \frac{g\, m_2\, s^2 + g\, k}{m_1\, m_2\, s^4 + ((k+g)\ m_2 + k\, m_1)\ s^2 + g\, k} \tag{27}$$