1 FEM formulation

1.1 Poisson equation

Let us start our discussion about the FEM with the strong form of the Poisson's equation

$$\Delta T = f(x), \quad x \in \Omega, \tag{1}$$

$$T = u(x), \quad x \in \Gamma_D, \tag{2}$$

$$\nabla T \cdot \mathbf{n} = g(x), \quad x \in \Gamma_N, \tag{3}$$

where $\Omega \subset \mathbb{R}^n$ is the solution domain with the boundary $\partial\Omega$, Γ_D is the part of the boundary were Dirichlet boundary conditions are given, Γ_N is the part of the boundary where Neumann boundary conditions are given, $T(x)$ is the unknown function to be found, $f(x), u(x), g(x)$ are the known functions.

The FEM method is based on a weak formulation. The weak form of the equation (1) is

$$\int\limits_\Omega (\Delta T - f) \cdot s \,\mathrm{d}V = 0,$$

where s is the *test* function. Integrating this equation by parts

$$0 = \int\limits_\Omega (\Delta T - f) \cdot s \,\mathrm{d}V = \int\limits_\Omega \nabla \cdot (\nabla T) \cdot s \, dV - \int_\Omega f \cdot s \,\mathrm{d}V =$$

$$= -\int\limits_\Omega \nabla T \cdot \nabla s \,\mathrm{d}V + \int\limits_\Omega \nabla \cdot (\nabla T \cdot s) \,\mathrm{d}V - \int\limits_\Omega f \cdot s \,\mathrm{d}V$$

and applying Gauss theorem we obtain:

$$0 = -\int\limits_\Omega \nabla T \cdot \nabla s \,\mathrm{d}V + \int\limits_{\Gamma_D \cup \Gamma_N} s \cdot (\nabla T \cdot \mathbf{n}) \,\mathrm{d}s - \int\limits_\Omega f \cdot s \,\mathrm{d}V$$

or

$$\int\limits_\Omega \nabla T \cdot \nabla s \,\mathrm{d}V = \int\limits_{\Gamma_D \cup \Gamma_N} s \cdot (\nabla T \cdot \mathbf{n}) \,\mathrm{d}s - \int\limits_\Omega f \cdot s \,\mathrm{d}V$$

The surface integral term can be splitted into two integrals, one over the Dirichlet part of the surface and second over the Neumann part

$$\int\limits_\Omega \nabla T \cdot \nabla s \,\mathrm{d}V = \int\limits_{\Gamma_D} s \cdot (\nabla T \cdot \mathbf{n}) \,\mathrm{d}s + \int\limits_{\Gamma_N} s \cdot (\nabla T \cdot \mathbf{n}) \,\mathrm{d}s - \int\limits_\Omega f \cdot s \,\mathrm{d}V \tag{4}$$

The equation (4) is the weak form of the equation (1). But we can not solve it without applying the boundary conditions. So it is a time to talk about the boundary conditions.

1.2 Dirichlet Boundary Conditions

On Dirichlet part of the surface we have two restrictions. One is the Dirichlet boundary conditions $T(x) = u(x)$ as it is, and the second is the integral term over Γ_D in equation (4). To be consistent we have to use only the Dirichlet conditions and avoid the integral term. To implement it we can take the function $T \in V(\overline{\Omega})$ and the test function $s \in V_0(\overline{\Omega})$, where

$$V(\overline{\Omega}) = \{f(x) \in H^1(\overline{\Omega})\},$$

$$V_0(\overline{\Omega}) = \{f(x) \in H^1(\Omega); f(x) = 0, x \in \Gamma_D\}$$

In other words the unknown function T must be continues together with its gradient in the domain and on the surface. In contrast the test function s must be also continues together with its gradient in the domain but on the surface Γ_D it is defined to be zero.

With this requirements the integral term over the Dirichlet part of the surface is vanishing and the weak form of the Poisson equation becomes

$$\int\limits_{\Omega} \nabla T \cdot \nabla s \, \mathrm{d}V = \int\limits_{\Gamma_N} s \cdot (\nabla T \cdot \mathbf{n}) \, \mathrm{d}s - \int\limits_{\Omega} f \cdot s \, \mathrm{d}V, \quad s \in V_0(\overline{\Omega}),$$

$$T(x) = u(x), \quad T \in V(\overline{\Omega}).$$

That is why the Dirichlet conditions in FEM terminology are called as *Essential Boundary Conditions*. These conditions are not a part of the weak form and they are used as it is.

Practically in numerical implementation of the FEM the Dirichlet boundary conditions are used as following:

1. First, the system of linear equations are assembled using the all terms in weak form.
2. Next, the known Dirichlet values are used to substitute the corresponding row/column in the system of linear equations.

1.3 Neumann Boundary Conditions

The Neumann boundary conditions are the known flux $\nabla T \cdot \mathbf{n} = g(x)$. The integral term over the Neumann surface in the equation (4) contains the same flux. So we can use the known function $g(x)$ in the integral term:

$$\int\limits_{\Omega} \nabla T \cdot \nabla s \, \mathrm{d}V = \int\limits_{\Gamma_N} s \cdot g \, \mathrm{d}s - \int\limits_{\Omega} f \cdot s \, \mathrm{d}V, \quad s \in V_0(\overline{\Omega}),$$

That is why the Neumann conditions in FEM terminology are called as *Natural Boundary Conditions*. These conditions are a part of weak form terms.

1.4 The weak form of the Poisson equation

Now we can write the resulting weak form for the Poisson equation (1)–(3)

$$\int_{\Omega} \nabla T \cdot \nabla s \, \mathrm{d}V = \int_{\Gamma_N} s \cdot g \, \mathrm{d}s - \int_{\Omega} f \cdot s \, \mathrm{d}V, \quad s \in V_0(\overline{\Omega}), \tag{5}$$

$$T(x) = u(x), \quad T \in V(\overline{\Omega}). \tag{6}$$

2 Numerical solution of the problem

To solve numerically the given problem using the FEM we have to go through 5 steps

1. Define the geometry of the domain and surface.
2. Define the known functions.
3. Define the unknown and test functions.
4. Define essential boundary conditions (Dirichlet conditions).
5. Define the equation and the natural boundary conditions (Neumann conditions) as a set of the integral terms.

In SfePy the geometry is defined in the “region” section of the problem file, for example

```
region_1000 = {
    'name' : 'Omega',
    'select' : 'elements of group 6',
}

region_03 = {
    'name' : 'Gamma_Left',
    'select' : 'nodes in (x < 0.00001)',
}
```

The known function is defined in the “material” section of the problem file, for example

```
material_2 = {
    'name' : 'coef',
    'values' : {'val' : 1.0},
}
```

The unknown and test function is define in "field" and "variables" section of the problem file, for example

```
field_1 = {
    'name' : 'temperature',
    'dtype' : 'real',
    'shape' : (1,),
    'region' : 'Omega',
    'approx_order' : 1,
}

variable_1 = {
    'name' : 't',
    'kind' : 'unknown field',
    'field' : 'temperature',
    'order' : 0, # order in the global vector of unknowns
}

variable_2 = {
    'name' : 's',
    'kind' : 'test field',
    'field' : 'temperature',
    'dual' : 't',
}
```

The essential (Dirichlet) boundary conditions are defined in "ebc" sections of the problem file, for example:

```
ebc_1 = {
    'name' : 't1',
    'region' : 'Gamma_Left',
    'dofs' : {'t.0' : 2.0},
}

ebc_2 = {
    'name' : 't2',
    'region' : 'Gamma_Right',
```

```
    'dofs' : {'t.0' : -2.0},
}
```

The equation and natural (Neumann) boundary conditions are defined as a set of integral terms in the “integral” and “equation” sections of the problem file, for example:

```
integral_1 = {
    'name' : 'i1',
    'kind' : 'v',
    'order' : 2,
}

equations = {
    'Temperature' : """dw_laplace.i1.Omega( coef.val, s, t ) = 0"""
}
```

3 Examples

3.1 diffusion/poisson.py

Let us start with the first very simple example

$$c \cdot \Delta T = 0, \quad x \in \Omega,$$

$$T(x) = 2.0, \quad x \in \Gamma_L,$$
$$T(x) = -2.0, \quad x \in \Gamma_R,$$
$$\nabla T \cdot \mathbf{n} = 0, \quad x \in \partial\Omega \setminus (\Gamma_L \cup \Gamma_R),$$

where the c is the known constant.

Compare this equation with the full Poisson problem (1)–(3) we can see, that the $f(x) = 0$ and the $g(x) = 0$. Therefore the weak form for this equation is

$$\int_\Omega c \cdot \nabla T \cdot \nabla s \, \mathrm{d}V = 0, \quad s \in V_0(\overline{\Omega}),$$

$$T(x) = 2.0, \quad x \in \Gamma_L,$$
$$T(x) = -2.0, \quad x \in \Gamma_R.$$